BUNVIS-RG: Exact Frame Buckling and Vibration Program, with Repetitive Geometry and Substructuring

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BUNVIS-RG is a computationally efficient FORTRAN 77 program which uses an exact stiffness matrix method to find the eigenvalues (i.e., critical load factors, or natural frequencies of vibration) and modes of three-dimensional frames without making the usual approximations of conventional finite element analyses. BUNVIS-RG includes: axially loaded Timoshenko or Bernoulli-Euler beams with distributed mass; taut strings; tapered and other nonuniform members; very efficient computation of modal densities; initial static calculations which include prestress and acceleration; dead and live load in buckling problems; rigid body modes; bandwidth reduction; treatment of stayed columns as substructures; treatment of linearly and rotationally periodic structures by analyzing a single repeating element; and elastic and offset connections between members and nodes. These and other features of BUNVIS-RG are described in this paper, which also gives results to illustrate its scope and efficiency and to establish a formula for predicting solution times.

I. Introduction

BUNVIS-RG (BUckling or Natural VIbration of Space frames with Repetitive Geometry) is a FORTRAN 77 computer program with approximately 11,000 lines of coding, which uses the stiffness matrix method and exact member theory. It finds the eigenvalues, i.e., the critical load factors in buckling problems or the natural frequencies in undamped vibration problems, with the option of finding the corresponding modes. It covers three dimensional frames consisting of uniform beam-column members and/or taut strings. Special care has been taken to account for rigid body freedoms and prestressing, so that frames in space can be analyzed. An optional static analysis provides the internal member forces that are needed by the eigenvalue analysis. The eigenvalue results are exact in the sense that the beam-column member equations used were obtained by solving the governing differential equations of Timoshenko theory to allow for the effects of axial force and shear deflections and, additionally in vibration problems, of distributed mass and distributed rotatory inertia. Dramatic computational savings can often be obtained by using the exact repetitive geometry and substructuring features of the program. The former enables frames which are repetitive in any number of coordinate directions, including rotational periodicity, to be treated by solving an eigenvalue problem of the size associated with the repeating portion of the frame. The capability of BUNVIS-RG is illustrated by Figs. 1–5, which are discussed in more detail later.

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BUNVIS-RG has been developed from the earlier program BUNVIS,[1] which was not made generally available although it has been used for several years. BUNVIS-RG was developed from BUNVIS by adding repetitive geometry and many practical features, such as the preliminary static calculations, automatic node renumbering to reduce bandwidth, a "user friendly" data preparation scheme, plotting, flexible joints, and offset connections at nodes. All such features are treated exactly for eigenvalue problems, whereas conventional finite-element programs usually make some approximations unless additional nodes are added.

BUNVIS-RG is currently available to U.S. users from COSMIC,‡ and to other users from the second author. The principal aims of the present paper are to describe the program's capabilities, to illustrate them with example problems, and to indicate the computer time and storage which it requires, in a clear and straightforward way which will enable practicing engineers to assess its relevance to their problems. To avoid obscuring these aims, the novel aspects of the theory, and fully detailed examples, are given in separate publications which are cross-referenced. Thus the examples in the paper have been chosen to illustrate the scope of the program, and to show VAX solution times. Based on these results, simple formulas are presented for estimating the solution time and storage required for any problem.

II. Main Features of BUNVIS-RG

BUNVIS-RG uses exact member theory (see Sec. IIA) for beams and stays which accounts for distributed mass and axial force correctly, and finds selected eigenvalues by using a theoretically established algorithm[2,3] which ensures that none are missed. The algorithm involves a stiffness matrix that is a function of the eigenparameter, i.e., of the frequency or load factor. Gauss elimination is applied to this matrix at a trial value of the eigenparameter to enable the algorithm to determine the number of eigenvalues exceeded, and this is repeated

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for successive trial values of the eigenparameter. In previous applications these trial values have often been chosen by a bisection routine. However, BUNVIS-RG uses a new accelerated convergence routine which is described in a separate paper[4] and which retains the certainty of the algorithm while being about twice as fast as bisection. The program can account for the effects of axial forces on the flexural vibrations of members, so that natural frequencies can be found for loaded frames. The axial forces can be given as data, or can be found from preliminary static calculations which include the combined effects of external loads applied at the nodes (i.e., joints), acceleration (e.g., gravity), and axial member preloads or prestrains. By using the simple data input scheme given in its User Manual,[5] BUNVIS-RG can allow for: any topology of a three-dimensional frame; rotatory inertia and shear deflection of members; unequal principal second moments of area about arbitrarily oriented principal axes of cross sections of members; generally tapered members with varying axial load; elastic connections between members and nodes; offset connections between members and nodes; cylindrical or Cartesian coordinates; and nodes which have lumped masses and/or elastic or rigid supports. In addition, BUNVIS-RG can: find buckling or vibration modes; plot such modes and the undeflected shape of the frame; give greatly reduced solution times for structures which are linearly and/or rotationally repetitive; and include stayed columns in a frame by using a very efficient substructuring capability. The remaining subsections of this section give detail and references where such are needed to define the theoretical aspects of the above features. An important contrast with finite-element programs is that exactness is retained by all these features, except for some of the static calculations, and except for the most general of the tapered member options, in which the tapered member is approximated by a user-selected number of uniform members which are assembled by an exact substructure procedure. Otherwise, there is no need to introduce extra nodes or members even for the highest eigenvalues because the member theory retains the infinite number of degrees of freedom, and hence of eigenvalues, of the real frame.

A. Member Theory

The member equations used are the classically exact ones obtained by solving the appropriate differential equations. The stiffness coefficients which result are functions of the axial force in the member, and are also functions of frequency in vibration problems. The expressions used for the stiffness coefficients of a taut string are those given in Ref. 6. The remaining stiffness coefficients are essentially refinements of those given in Ref. 7 for vibrations of an axially loaded Timoshenko beam. These have the data-triggered option of omitting any desired combination of frequency, axial force, rotatory inertia, and shear deflection. Thus static and Bernoulli-Euler stiffnesses are included.

B. Convergence on Eigenvalues

BUNVIS-RG can find: a set of data-specified eigenvalues, such as the first and the third, where the first is the fundamental; all eigenvalues between specified values of the eigenparameter, e.g., to find all the natural frequencies in a frequency band; or the number of eigenvalues in each of the equal intervals between evenly spaced values of the eigenparameter, e.g., enabling modal densities to be found. These options all use the theoretically proven algorithm described above and so they cannot miss any natural frequencies of (dead) loaded frames,[2] or any critical load factors of frames subjected to a dead (i.e., constant) load system and to a live load system which is scaled by the load factor[3] (i.e., proportional loading). The simplest route in BUNVIS-RG assumes that the dead or live load systems can be approximated adequately by the axial forces which they cause in the members, and that these axial forces are known a priori and constitute the dead and live load data. The alternative routes involve the static calculations that follow.

C. Static Loading, Including Prestress and Acceleration

For buckling and vibration problems, BUNVIS-RG can calculate the dead load axial forces caused in the members by any combination of externally applied static point forces and moments at the nodes, axial preload or prestrain in the members, and acceleration loads such as those due to gravity. This is done by replacing the preloads, prestrains, and acceleration forces by equivalent static point forces at the nodes, adding these to the externally applied forces at the nodes, and then solving by the usual static stiffness matrix method. The live load axial forces can be found similarly, except that acceleration is not included.

Such preload and prestrain calculations in effect assume that the frame is assembled with the preloads and prestrains in its members given by data and its nodes clamped, and that the nodes are then released to distribute the prestress through the structure. Gravity, or any other acceleration loading of a structure, is accounted for by sharing the mass of each member or substructure between the nodes at its ends, according to its center of gravity location, and converting to forces by using the appropriate acceleration. The acceleration is specified by giving its components in the three principal coordinate directions, so that linear acceleration in any direction can be treated.

The static ~~stiffness matrix is~~ a function of the initial axial forces in the members, which are supplied as data. When dead load and live load axial forces are given in data they are added together with the latter scaled by the initial load factor. Hence the most accurate results are obtained when these axial forces are close approximations to the final axial forces in the members.

The static loading calculations should alter the dead and live load axial forces in members within a stayed column substructure from those given in data, but at present they do not do so. (However the end load on the stayed column is calculated and could be used to find the axial forces in its members, e.g., by a separate BUNVIS-RG run.)

D. Dead Load and Rigid Body Modes

It is meaningless to find natural frequencies of a buckled frame, and similarly the algorithm used[3] to find critical buckling loads of frames subjected to dead and live loading presupposes that the frame is stable when the dead load is applied alone. Therefore for both vibration and buckling problems, a preliminary calculation is performed which uses the algorithms of Refs. 2 and 3 to check that the frame is stable under the dead load alone. This check can fail when rigid body freedoms exist, unless precautions are taken. Therefore BUNVIS-RG allows a single data-specified small stiffness to be added to all three translational freedoms at every node of the frame. Suitable choice of this stiffness leaves the eigenvalues sensibly unaltered, while replacing the rigid body modes by very low eigenvalues which need not be computed. However, the presence of even quite small implied external loads at the nodes must then be avoided, because they could make this frame unstable. Such loads would result if the dead load axial forces are not close enough to being in equilibrium at the nodes. This happens if the axial forces used are those given in data, and the forces and/or geometry are not given to high accuracy. It would also usually happen if the axial forces are calculated by the program (see the preceding section) because such calculations include the shear forces, so that the axial forces alone will not usually be in equilibrium at the nodes. To avoid this difficulty the program has three options which may be employed for structures that have rigid body degrees of freedom. The choice depends on the type of structure. For structures that are not mechanisms when modeled as pin-connected trusses, the static analysis may be performed on this basis to yield axial forces that are in equilibrium. If the user is satisfied that these forces differ very little from those found using the full member stiffnesses, the error in the subsequent eigenvalue calculation based on the full member stiffnesses should be small. If this approach is not suitable but shear forces from the stress analysis are reasonably small, an alternative approach is to specify a small quantity (PDELTA) in the input which increases all tension loads, and decreases all compression loads, by PDELTA times their correct values. If a value of PDELTA can be found that will result in a net tension across the structure, no instability will occur. The error in any eigenvalue calculated in this way, when expressed as a ratio to the correct eigenvalue, is usually less than PDELTA. As a final option, if the static shear forces are large the user can allow for them in the eigenvalue analysis, although the method involved is approximate and must be used with care, see Ref. 5.

E. Model Finding

Modes are found by a random force vector method. This method is the "P_{RT} method" which was advocated and evaluated in Ref. 9. Briefly stated, if K, D, and P are, respectively, the stiffness matrix of the frame and its displacement and force vectors, the P_{RT} method consists of solving the equation

$$KD=P \tag{1}$$

with the elements of P given random values and with K evaluated at a close approximation to the eigenvalue. When several eigenvalues are coincident, the program finds a mode for each by the expedient of using a different random force vector in each case.

F. Use of Core Storage and Resequencing of Nodes

BUNVIS-RG is an in-core program (except for data handling, plotting, and, optionally, for mode finding and the static loading calculations of Sec. IIC), which was written to keep the amount of coding and the core storage requirement, for data and working space combined, as small as was reasonably possible while also keeping execution time near optimum. Thus it employs a fixed bandwidth method which only stores the active triangle of the band during assembly and Gauss elimination, as described in connection with Fig. 2 of Ref. 10. This results in low working space requirements, but if modes are required the entire upper half band of the stiffness matrix must be stored. The user has the option of doing this in core or in backing storage, depending on problem size and storage available. Virtually all the array space needed is contained by three one-dimensional arrays, which store the integer, real and complex numbers of the data and working space without gaps. Working space is minimized by reusing it whenever possible.

Because solution time and storage are bandwidth dependent, BUNVIS-RG incorporates the program BANDIT[11] to give the option of reducing the bandwidth. BANDIT uses both the Cuthill-McKee[12] method and the Gibbs-Poole-Stockmeyer[13] method. Its four options for resequencing[11-13] are retained and use the following criteria: 1) rms wavefront,

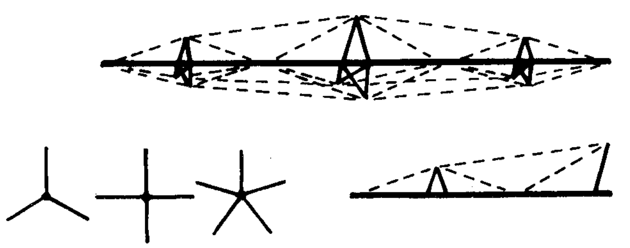


Fig. 1 Typical stayed column with N stays frames: —, core; —, beams; ---, stays. a) Column, with $N=3$; b) end views, for $N=3$, 4, or 5; c) representative half frame.

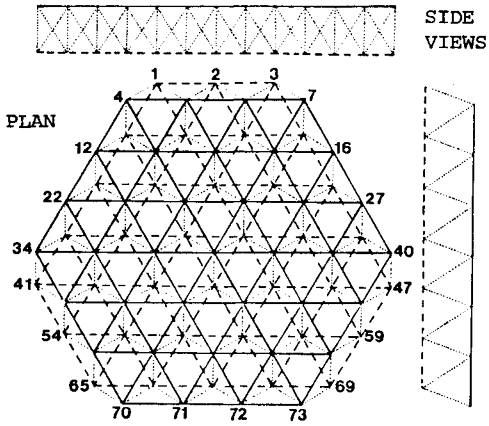


Fig. 2 Tetrahedral truss analyzed. Lines denote members: —, top plane; ---, bottom plane; · · ·, interplane. There are 276 members, and the 73 nodes have 73×6=438 degrees of freedom and are numbered in horizontal rows as indicated.

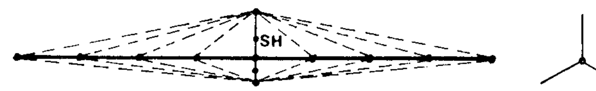


Fig. 3 Side and end views of one of the set of 276 identical stayed columns used in place of the beam-column members of the tetrahedral truss of Fig. 2: —, core; —, spokes; ---, stays; •, point mass; SH, spoke hinge.

2) bandwidth, 3) profile, and 4) maximum wavefront. Option 2, which aims to minimize bandwidth, is the default option in BUNVIS-RG.

G. Substructuring Method for Stayed Columns

An important feature of BUNVIS-RG, which accounts for about 250 lines of coding, is the inclusion of a very efficient (i.e., low storage requirement and fast execution), *exact*, substitute column method[14] for handling stayed columns as substructures.¶ These stayed columns, see Fig. 1, are symmetric about their mid-length and consist of a central core with $N=3,4,5,\ldots$ identical stay frames equally spaced around it. The stay frames can consist of any combination of beam-columns and taut strings (i.e., stays) which lie in a plane, but stay frames are not interconnected except where they are connected to the core. Thus Fig. 1a shows a typical stayed column, which can be represented in data merely by the representative half frame of Fig. 1c.

There is no need to present the detailed data (e.g., the flexural rigidities of members) when giving examples of solution times and storage requirements, but it is necessary to indicate where members were of the same type or substructures were identical, etc. A published example[15] is that the tetrahedral truss of Fig. 2 was analyzed with its 276 members all identical and uniform, and was reanalyzed with the members replaced by 276 identical stayed columns of the type shown in Fig. 3. The efficiency of the substructuring route is such that the reanalysis only added about one percent to the computer time needed to find one eigenvalue. Most of the eigenvalues occurred in large narrowly banded groups when the stayed columns were used, and hence the frequency range of interest contained 4,978 natural frequencies. This structure has 21,966 degrees of

¶Reference 14 gives several variants of the substitute column method. The one used in BUNVIS-RG is particularly efficient, being that described around Fig. 3,[14] in the form in which three degrees of freedom are allocated to every node and are chosen to permit whichever of in-plane or out-of-plane deflections of each stay frame of the type I, II_a, or II_b substitute columns is required by the theory.

freedom at its nodes if substructure nodes are included but, because the program can bracket eigenvalues without finding them individually, all 4,978 were found to reasonable accuracy in about two hours of VAX-11/780 CPU time.[15] However, it is more realistic to quote about 7 min per well-separated eigenvalue. These times were calculated by using an adaptation of the earlier program BUNVIS. It is estimated that BUNVIS-RG would take about two-thirds of this time, because of the accelerated convergence routine mentioned at the beginning of section II, so long as the real arithmetic version (see section II H. below) of the program is used.

H. Repetitive Geometry

Frames which contain a group of nodes and members which repeats in one or more coordinate directions may be analyzed using the repetitive feature of the program. The size of the problem analyzed is only that corresponding to the repeating portion of the frame but results are applicable to the complete frame. Figures 4 and 5 illustrate large repetitive structures that have been analyzed by using this repetitive geometry feature. The truss of Fig. 2 has also been analyzed by using this feature (see Fig. 7 and Table 2), because it consists of three portions which are identical except that each is rotated by $2\pi/3$ relative to the others. In general BUNVIS-RG handles linear repetition in any combination of one, two (see Fig. 4), or three Cartesian directions, and also handles rotational periodicity (see Fig. 5) with the option of linear repetition in the direction of the axis of rotation.

The detail of the repetitive geometry method is given in full in Ref. 16, with the central node theory of its Appendix replaced by the alternative and more general approach of Ref. 17. Briefly, this method requires that the response mode be repetitive over a certain number of bays in each coordinate direction, as given by Eq. (2) of Ref. 16 as

$$D_j = D_0 \exp\{2i\pi(n_1 j_1/N_1 + n_2 j_2/N_2 + n_3 j_3/N_3)\} \quad (2)$$

where D_j is the displacement vector in the jth repeating portion located j_k bays in each coordinate direction from the structure actually modeled. The mode repeats in each coordinate direction in N_k bays. The n_k are harmonics of response which must cover the range

$$n_k = 0, \pm 1, \pm 2, \ldots \pm \text{integer}(N_k/2) \quad (3)$$

to ensure that all possible independent responses are accounted for. For structures repetitive in only one direction, negative values of n_k need not be considered. For structures repetitive in two or three directions, negative values of n_k need not be considered for one direction, but both positive and negative values must be used for the remaining directions. A more complete discussion of these requirements is given in Ref. 16.

Introduction of Eq. (2) results in complex quantities which are eventually incorporated into the stiffness matrix in the analysis. The program uses the complex arithmetic feature of Fortran to implement the solution. For nonrepetitive frames, the stiffness matrix is real. To account for this, it is possible to compile two versions of the program, a real arithmetic version for regular problems and repetitive problems with a harmonic response of 0, or $N_k/2$ when N_k is even, and a complex arithmetic version for general repetitive problems. The complex arithmetic version will work for the regular problem, but with a factor of approximately 2 penalty in storage compared to the real version, and a factor on solution time that can exceed 5 (see Sec. III). The only difference between the two versions is the removal of one line of code (identified with a comment card in the source code) declaring certain variables to be complex.

The program is particularly useful for frames having rotational periodicity, where the mode inevitably repeats in the circumferential direction and the results obtained are exact. Alternatively, for frames of finite size which are repetitive in rectangular coordinates and which have simply supported boundaries and appropriate symmetry in geometry and loading, exact solutions can be obtained by assuming a mode that is repetitive over twice the length of the actual frame. Moreover, even if symmetry and boundary condition requirements are not satisfied, so that the assumed mode shape will not be compatible with the actual supports, the results for wavelengths that are small relative to the length of the structure may still help the analyst.

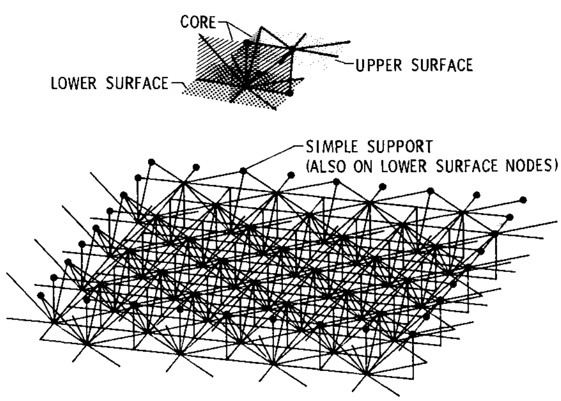


Fig. 4 Simply supported hexahedral truss platform, illustrating linear repetition in two directions. The repeating portion shown has only four nodes, compared with 162 for the full frame.

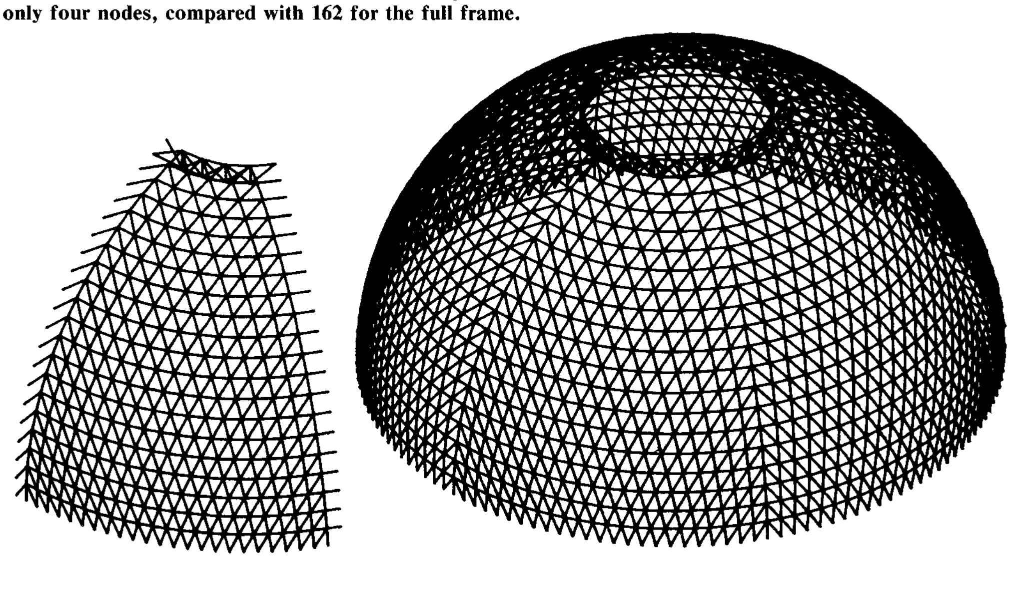
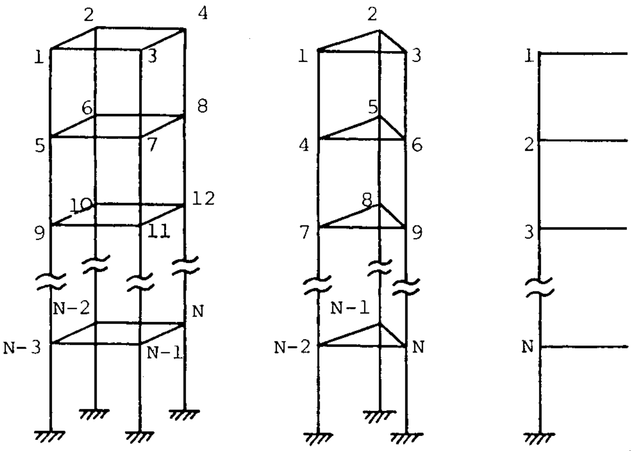


Fig. 5 Dome structure with stiffened hole, illustrating rotational periodicity. The repeating portion and the full frame have, respectively, 298 and 1788 nodes.



a) $N=4N_s$, $B=5$, $M=8N_s$ b) $N=3N_s$, $B=4$, $M=6N_s$ c) $N=N_s$, $B=2$, $M=2N_s$

Fig. 6 Unbraced single bay frames with N_s storeys: a) square plan form; b) equilateral triangular plan form; c) repeating portion needed when rotational periodicity is used to obtain results for the frames of a) or b).

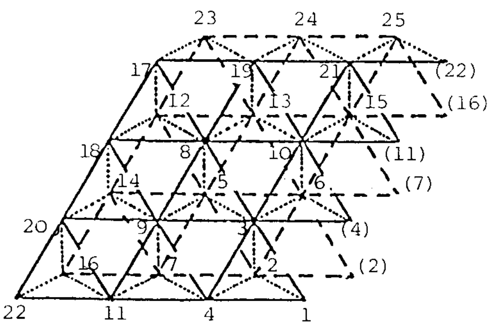


Fig. 7 Repeating portion for frame of Fig. 2. Node numbers in brackets are for the adjacent repeating portion and node 1 is on the axis of periodicity.

The computer time savings which can be obtained by using repetition are demonstrated more fully in Section III. However it is apparent that these savings, and savings of computer storage, must be very great simply by comparing the number of nodes for the full frames of Figs. 4 and 5 with the number of nodes for their repeating portions, since only the repeating portions are used by the repetitive analysis. The inclusion of rotationally periodic space frames with members along their axis of periodicity needed an extension of the original algorithm for ensuring convergence on all required eigenvalues, which Ref. 17 shows is valid for any rotationally periodic structure.

I. Elastic and Offset Connections Between Members and Nodes

By default, the program assumes that the centerlines of members pass through any node to which they are connected and that all six degrees of freedom (three translations and three rotations) are rigidly connected to the node. Alternatively, any combination of the six freedoms (in member coordinates) at each end of a member can be elastically connected to the node. Reference 18 gives details of the simple and efficient way in which these elastic connections are introduced one at a time so as to retain the exactness of the results and the certainty of the algorithm which ensures that no eigenvalues are missed. Zero elastic stiffnesses can be used to obtain pinned or sliding connections. Offsets, i.e., eccentric connections between a member and a node, can also be applied, using a standard transformation, in any combination of the three Cartesian global axis directions. Alternatively, the offsets can be defined using a

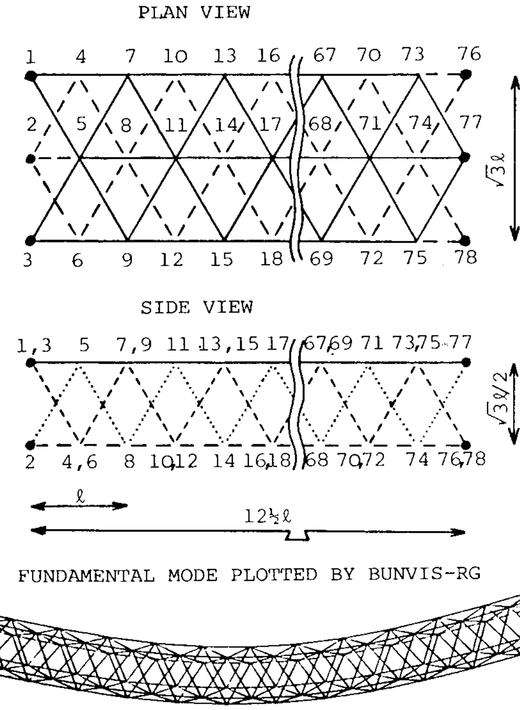


Fig. 8 Truss type girder composed of 247 identical, uniform, Bernoulli-Euler beams of length ℓ, flexural rigidity (for both axes) EI, mass per unit length μ, $EA=2.5\times10^5 EI/\ell^2$, $GJ=0.8\,EI$ and polar inertia per unit length $=2\mu\ EI/EA$. (Hence the slenderness ratio, ℓ/r, $=\ell\sqrt{EA/EI}=500$). • denotes a supported node.

local set of member axes. Note that when elastic connections and offsets are both present the former are applied to the member first. Thus, if the offsets are visualized as rigid links between the ends of the members and the nodes, the elastic connections are between the members and the rigid links, which are always rigidly connected to the nodes.

J. Tapered, Stepped, and Nonuniform Members

BUNVIS-RG includes exact Bernoulli-Euler theory member equations for tapered members which have taper such that their area and second moment of area vary according to, respectively,

$$A=A_0\left(1+c\frac{x}{L}\right)^n \quad \text{and} \quad I=I_0\left(1+c\frac{x}{L}\right)^{n+2} \tag{4}$$

with $n=1$ or 2, where x is measured from one end of a member of length L. The expressions used for the individual member stiffnesses are explicit ones which involve Bessel functions. They only cover either vibration of members which do not carry axial load[19] or static behavior of axially loaded members.[20] Thus they can be used for vibrating frames in which the tapered members are unloaded, or for buckling of frames.

BUNVIS-RG also includes a substructuring route which automatically generates a stepped beam which can represent a nonuniform member as any required number of rigidly jointed uniform members. These uniform members are analyzed by the Timoshenko beam-column theory described earlier in this paper, and their principal properties can be varied independently of each other between successive uniform members to follow profiles along the length of the nonuniform member which are given in data as algebraic curves. The data required by the program is very concise, and an extra data input refinement makes it even more so for profiles which satisfy Eq. (4). Thus tapered members which satisfy Eq. (4), but do not satisfy the requirements stated in

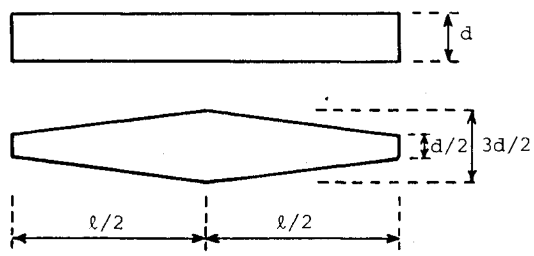


Fig. 9 Uniform thin hollow circular cross-section member for frame of Fig. 8, and the symmetrically linearly tapered member which shares its wall thickness t and its total mass. Note that since the slenderness ratio = 500 (see Fig. 8), $\ell d \simeq = 125\sqrt{2}$.

the paragraph which contains it (e.g., the beam is an axially loaded vibrating one, or $n > 2$, or Timoshenko theory is needed), require only a minimum of input data.

The substructuring facility also allows any number of uniform Timoshenko beam-columns and/or any combination of tapered or nonuniform members of the kinds already described in this section, to be rigidly or elastically connected in a straight line to form any tapered, stepped, or generally nonuniform beam-column for which flexure in the two principal planes, axial response, and torsional response are all uncoupled. Such uncoupling is also assumed throughout the analysis described earlier in this section, so that all substructuring uses minimum bandwidth substructure stiffness matrices by considering separately in-plane flexure in each of the two principal planes, axial behavior and torsional behavior.

III. Estimating Solution Times

T, the solution time to find eigenvalues (without modes or preliminary stress calculations), is almost exactly equal to the product of the number of iterations taken by the convergence routine, I, and the time taken per iteration, T_i. Thus

$$T \simeq I \times T_i \tag{5}$$

The value of I depends on the problem and can only be estimated from experience. Thus I typically depends upon the number of eigenvalues sought, the accuracy to which they are required, and whether they occur in clusters of close or coincident values. The convergence routines in BUNVIS-RG revert to bisection when finding coincident eigenvalues, but otherwise the accelerated convergence routine[4] usually uses far fewer iterations. As a rough guide, when eigenvalues are required to 0.001% accuracy the contributions received by I will be about sixteen for each group of coincident eigenvalues and about eight for each well-separated eigenvalue. If modes are computed in-core, the extra time taken can be estimated by increasing I by between 1 and 2 per mode. However, the accuracy of a mode is frequently much less than that of its eigenvalue, so that extra iterations may be needed to improve the accuracy of the mode by finding the eigenvalue more accurately.

T_i depends upon the problem and upon which of the real or complex arithmetic versions of BUNVIS-RG is used. It can be estimated as follows. The time taken by the fixed bandwidth Gauss elimination step of BUNVIS-RG is assumed to be proportional to $B^2(N-2B/3)$, where N is the number of nodes and B is one greater than the maximum node number difference of any pair of connected nodes, e.g., see Fig. 6. This time dominates T_i when B is large, but otherwise the time taken to compute member stiffnesses, and to transform them to global axes, is significant. These member calculations are performed almost entirely in real arithmetic in the complex arithmetic version of BUNVIS-RG, and so take nearly the same time whichever version is used. Hence (in the absence of stayed columns and tapered, stepped or nonuniform members) T_i can be estimated from

Table 1 Values of α and β for VAX-11/780

Real BUNVIS-RG		Complex BUNVIS-RG	
α	β	α	β
0.00164	0.0145	0.00843	0.0157

$$T_i = \alpha B^2 \left(N - \frac{2}{3}B\right) + \beta M \tag{6}$$

where α and β are computer and compiler dependent constants and M is the number of uniform beam-columns in the frame. Note that M includes all connections because identical members are treated separately in the simplest applications of the program. It is shown below that, for the VAX-11/780 computer (with floating point accelerator), T_i is given in CPU seconds by using Eq. (7) with the values of α and β given in Table 1.** These values were deduced from measured VAX iteration times. These times typically varied by a few percent between successive iterations. Therefore all times used were obtained by averaging for several (usually five) iterations. BUNVIS-RG permits multiple connections between the same pair of nodes, and so the values of β given in Table 1 were calculated as one-hundredth of the increase in iteration time that resulted from taking a two-node frame, which consisted of two randomly positioned nodes connected by a single member, and adding a hundred identical members to form 101 connections between the same two nodes. This value of β was checked by repeating for 50 and 150 additional members.

The VAX iteration times given in Table 2 were used to estimate α and to check the accuracy of Eq. (6). The results for the three frames of Fig. 6 were obtained for the separate cases of 2, 6, and 10 storeys, using the node numbering shown. The remaining results in the table are for the complete tetrahedral truss and node numbering of Fig. 2, and for the corresponding repetitive geometry analysis, using the repeating portion and node numbering of Fig. 7. Note that the real or complex arithmetic versions of BUNVIS-RG can be used to solve each of the problems except that only the complex version can be used when the repetitive geometry method is being used (i.e., for the frames of Figs. 6c and 7) unless, in Eq. (3), either $n_k = 0$ or $n_k = N_k/2$ when N_k is even. The results for the ten-storey square plan frame (i.e., the frame of Fig. 6a with $N_s = 10$) were used to obtain the values of α given in Table 1. The remaining results in Table 2 indicate the probable accuracy of Eq. (6) as a predictor of iteration times for frames, noting that the part of the predicted time contributed by the Gauss elimination related component of Eq. (6) (i.e., Δ) varies between 7% and 97%, and that for all sizable problems the measured error, ϵ, varies between $\pm 7\%$.

The values of β given in Table 1 were calculated for vibration of an arbitrarily aligned uniform Timoshenko member with unequal flexural rigidities for its principal planes of bending. Other results show that this value of β can be reduced by about: 5% if the member axis coincides with a global axis; 10% if the flexural rigidities are equal; 43% if one end of the member is connected to an "Earth" node; and 30% if member stiffnesses (but not their transformation to global coordinates) are calculated only once for identical members, by representing the beam as a "one-element stepped beam" (see below). These reductions were almost identical for both the real and complex versions of BUNVIS-RG. Similarly, results show that

**Preliminary calculations indiate that for a CDC Cyber 175 these values of α and β can be divided by about 10 when using real arithemetic, and by a higher denominator when using complex authemetic.

β can be increased by about 115% if there are any offsets at either end of a member, and by between 25% and 60% if there are elastic connections, depending upon how many of the twelve degrees of freedom at the two ends are elastically connected.

It is probably not worth altering Eq. (6) to compensate for any of the above changes of β, but it may be wise to adjust for the increase in T_i caused by using stepped or nonuniform members. Such adjustments can be made by treating each such member as δ uniform members when computing M, and results indicate that δ should have the values $\delta \simeq 2.1$ for a tapered member for which Bessel functions give exact dynamic stiffnesses, and $\delta \simeq 1 + (n_e - 1)/2.4$ for stepped members composed of n_e uniform elements, with the 2.4 emphasizing the efficiency of the calculations described at the end of the preceding section.

Timing runs for plane frames showed that, to a good approximation, $\alpha/2$ should be used in place of α when Eq. (6) is applied to such frames. This is consistent with the fact that in-plane and out-of-plane behavior are decoupled, since such decoupling results in about one-half of the multipliers encountered during the Gauss elimination being zero, and the Gauss elimination routine used skips operations that involve zero multipliers.

IV. Storage Requirements

Because the storage consists almost entirely of three arrays (one integer, one real and one complex) that are repeatedly reused to save space, it is difficult to provide a formula to relate the core storage requirement to the parameters of any given problem. However, at the data-reading stage BUNVIS-RG calculates the storage required and terminates with a message stating the array sizes needed if the sizes provided are inadequate. Because storage is reused efficiently, core storage is less likely to be a difficulty than is solution time. Hence the storage available is only likely to be exceeded for large problems, and then the storage required is usually dominated by the complex array (which becomes a real array for the real arithmetic version of BUNVIS-RG). In turn, the complex array is mainly accounted for by the active triangle of the stiffness matrix unless modes are computed in core, in which case the entire half band of the stiffness matrix must also be stored. Hence the complex array size needed is mainly accounted for by $18B^2$ unless modes are calculated in core, in which case it is a little over $18B^2 + 36NB$.

V. Results, Including Elastic and Offset Connections and Taper

Figure 8 shows a rigidly jointed frame, consisting of 247 identical uniform members that form a truss type girder of length $12\frac{1}{2}\ell$. This girder is varied to investigate the separate effects of introducing elastic or offset connections, or substituting tapered members for the uniform ones.

Solid and long dashed lines on the plan view of Fig. 8 represent members which, respectively, lie in an upper and a lower horizontal plane. On the side view the dotted lines represent members lying in the longitudinal vertical plane of symmetry and the short dashed lines represent members which lie in both of the two symmetrically situated longitudinal vertical planes which are clearly identifiable from the node numbering shown on the plan and side views. Figure 8 also shows the fundamental mode of vibration, as calculated and plotted by BUNVIS-RG. The properties of the members are shown in the caption, and the supports at each of the six nodes shown as solid circles are simple supports, in the sense that they leave all three rotational freedoms, and longitudinal translation, unrestrained while preventing translation in the plane containing nodes 1, 2, and 3, and the plane of nodes 76, 77, and 78. There are no other supports, the only mass is that of the members and the frame is unloaded.

If the single (longitudinal) rigid body freedom is ignored, the fundamental frequency corresponds to the "overall" mode shown in Fig. 8, which is symmetric about the vertical longitudinal plane of symmetry of the frame. Thus if the entire frame is thought of as forming a simply supported/simply supported beam of length $12\frac{1}{2}\ell$, breadth $\sqrt{3}\ell$ and depth $\sqrt{3}\ell/2$, the mode is a single half wave of length $12\frac{1}{2}\ell$. The corresponding natural frequency of $1.107\{EI/(\mu\ell^4)\}^{\frac{1}{2}}$ Hz is given as the basic case of Table 3. The table also shows how this frequency is altered by introducing either elastic or pinned connections at both ends of every member, or axial offsets (representing rigid gussets at joints) at both ends of every member, or the tapered members of Fig. 9 instead of uniform ones. The elastic connections resist flexural type relative rotations of the end of the member and the node, for all planes of flexure, with stiffness $k = EI/\ell$. The remaining freedoms are rigidly connected, i.e., the torsional rotation and all three translation components at the end of the member are shared by the node. The pinned connections are identical to the elastic ones, except that the stiffness $k = EI/\ell$ is replaced by zero, and BUNVIS-RG gave a mode which is antisymmetric about the vertical longitudinal plane of symmetry of the frame. The axial offsets were of length 0.1ℓ, so that the member length is reduced to 0.8ℓ (i.e., the member consists of a flexible member

Table 2 Comparison of predicted and measured times per iteration for a VAX-11/780 computer: ϵ is the percentage by which Eq. (6) overestimates measured time and Δ is the percentage of the predicted time related to Gauss elimination [i.e., contributed by the first of the two terms which are added in Eq. (6)]

Frame						VAX-11/780 CPU seconds per iteration							
						Real arithmetic				Complex arithmetic			
Fig. no.	Brief description	N_s	N	B	M	Eq. (6)	VAX run	ϵ, %	Δ, %	Eq. (6)	VAX run	ϵ, %	Δ, %
6a	Square	2	8	5	16	0.423	0.432	−2	45	1.235	1.282	−4	80
	plan	6	24	5	48	1.543	1.544	0	55	5.109	5.136	−1	85
		10	40	5	80	2.663	2.666	0	57	8.983	8.976	0	86
6b	Triangular	2	6	4	12	0.261	0.274	−5	33	0.638	0.688	−7	70
	plan	6	18	4	36	0.924	0.934	−1	44	2.633	2.650	−1	79
		10	30	4	60	1.587	1.620	−2	45	4.629	4.728	−2	80
6c	Repeating	2	2	2	4	0.062	0.078	−21	7	0.085	0.098	−13	26
	portion	6	6	2	12	0.205	0.224	−8	15	0.346	0.344	1	46
	for above	10	10	2	20	0.347	0.368	−6	16	0.606	0.586	3	48
2	Tetrahedral truss		73	15	276	27.25	26.56	3	85	123.8	123.3	0	97
7	Repeating portion		25	12	92	5.349	5.006	7	75	22.08	22.19	0	93

Table 3 Dimensionless fundamental natural frequencies for the frame of Fig. 8, and their variations, ϵ, caused by elastic connections, pin joints, offsets and member taper

Variant of problem	Freq.[a]	ϵ, %
Basic case	1.107	–
Basic case + elastic joints	1.015	−8
Basic case + pin joints	0.972	−12
Basic case + offsets	1.247	13
Basic case with taper	1.028	−7

[a]To be multiplied by $\{EI/(\mu\ell^4)\}^{1/2}$ to give Hz.

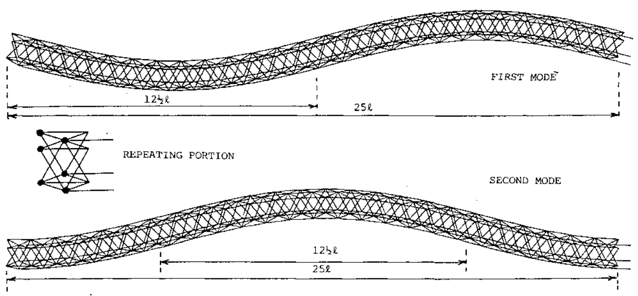


Fig. 10 The two fundamental modes plotted by BUNVIS-RG for the frame of Fig. 8, when the repeating portion shown is used. Note that only the six nodes denoted • belong to the repeating portion, because the other nodes at which the 20 members of the portion meet lie in the adjacent portion.

Table 4 Number of natural frequencies in bands of width $0.4\{EI/(\mu\ell^4)\}^{1/2}$Hz, for the frame of Fig. 8 and its variants

Variant of problem	Upper bound on frequency band $\times\{EI/(\mu\ell^4)\}^{1/2}$ gives Hz						
	1.2	1.6	2.0	2.4	2.4	3.2	3.6
Basic case	2	0	25	78	66	41	281
Basic case + elastic joints	2	1	490	1	1	0	2
Basic case + pin joints	2	491	0	1	2	1	2
Basic case + offsets	0	4	0	1	2	43	28
Basic case with taper	2	1	62	133	295	1	0

of length 0.8ℓ with rigid links of combined length 0.2ℓ at its ends), and so the mass per unit length was increased to 1.25μ as a convenient way of leaving the mass of the frame unaltered. The taper of one of the symmetric halves of the tapered member is given by Eq. (4) with $n=1$ and $c=2$, and the exact Bessel function stiffnesses were used.

It would be interesting to calculate the variation ϵ of Table 3 for a high natural frequency of the frame of Fig. 8. However, the bulk of the higher natural frequencies are closely grouped, so that it would be difficult to ensure that corresponding modes were being compared when calculating ϵ. Therefore, the modal density option of BUNVIS-RG has been used instead, to give Table 4.

The complete frame of Fig. 8 was analyzed to obtain the results of Tables 3 and 4 even though computer time and storage could be saved because the symmetry about the vertical longitudinal plane optionally permits only half of the frame to be analyzed by BUNVIS-RG, if due care[21] is taken. Alternatively, computer time and storage can be saved by using the six node repeating portion of Fig. 10 to obtain natural frequencies by the repetitive geometry method. These frequencies will not be exact duplicates of those obtained by analyzing the complete frame, because of detailed differences at the boundaries. For a smaller number of bays than the 12½ of Fig. 8 the frequencies might differ significantly, but for the frame of Fig. 8 the fundamental frequency agreed with the 1.107 of Table 3 to the accuracy given. This frequency was obtained by assuming repetition of the mode over a length 25ℓ, i.e., by using Eq. (2) with $N_1=25$, $n_1=1$, $n_2=n_3=0$ and $N_2=N_3=1$. BUNVIS-RG gives two modes whenever the exponential of Eq. (2) is complex. Figure 10 shows that, for the fundamental mode of the frame of Fig. 8, the two modes are identical except for a shift of $25\ell/4$ along the longitudinal axis. Thus it can be visualized that the required simply supported frame of Fig. 8 is approximated by taking the left-hand half of the upper mode, or the central half of the lower mode, as indicated by the dimensions of length $12\frac{1}{2}\ell$ shown.

VI. Concluding Remarks

The many features of the exact FORTRAN computer program BUNVIS-RG were summarized in the Introduction and the first paragraph of Sec. II. Principally, it finds the eigenvalues (i.e., critical load factors or natural frequencies) of three-dimensional frames without missing any and without making the usual approximations of conventional finite element programs. Hence there is no need for convergence checks, or for the introduction of extra elements when finding the higher natural frequencies of vibration. The capability of the program is explained in detail in Sec. II, which also cross-references the appropriate theory. Timing results for a substantial range of problems show that BUNVIS-RG solution times can be predicted reasonably accurately from two computer-dependent constants, the number of nodes and members in the frame, and the maximum node number difference of any pair of connected nodes. These and other results presented show that very large computational economies can result from using the repetitive geometry, the stayed column substructures, and the various nonuniform member options of BUNVIS-RG.

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